

COMPATIBLE METRICS ON A MANIFOLD AND NON-LOCAL BI-HAMILTONIAN STRUCTURES

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ABSTRACT. Given a flat metric one may generate a local Hamiltonian structure via the fundamental result of Dubrovin and Novikov. More generally, a flat pencil of metrics will generate a local bi-Hamiltonian structure, and with additional quasi-homogeneity conditions one obtains the structure of a Frobenius manifold. With appropriate curvature conditions one may define a curved pencil of compatible metrics and these give rise to an associated non-local bi-Hamiltonian structure. Specific examples include the F -manifolds of Hertling and Manin equipped with an invariant metric. In this paper the geometry supporting such compatible metrics is studied and interpreted in terms of a multiplication on the cotangent bundle. With additional quasi-homogeneity assumptions one arrives at a so-called weak \mathfrak{F} -manifold - a curved version of a Frobenius manifold (which is not, in general, an F -manifold). A submanifold theory is also developed.

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1. INTRODUCTION

Let M be a smooth manifold. The space of smooth vector fields on M will be denoted $\mathcal{X}(M)$ and the space of smooth 1-forms on M will be denoted $\mathcal{E}^1(M)$. If g is a (pseudo-Riemannian) metric on M , we shall denote by g^* the induced metric on T^*M . For a vector field X , $g(X)$ will denote the 1-form corresponding to X and for a 1-form α , $g^*\alpha$ will denote the vector field corresponding to α (via the isomorphism defined by g between TM and T^*M).

In this paper we will study the geometry induced by two metrics g and \tilde{g} on M . Unless otherwise stated, we will always denote by ∇ , R ($\tilde{\nabla}$, \tilde{R}) the Levi-Civita connection and the curvature tensor of g (\tilde{g} respectively). For every constant λ let $g_\lambda^* := g^* + \lambda\tilde{g}^*$, which, we will assume, will always be non-degenerate. The Levi-Civita connection and curvature tensor of g_λ will be denoted by ∇^λ and R^λ respectively.

Definition 1. [11]. *The metrics g and \tilde{g} are almost compatible if the relation*

$$(1) \quad g_\lambda^* \nabla_X^\lambda \alpha = g^* \nabla_X \alpha + \lambda \tilde{g}^* \tilde{\nabla}_X \alpha$$

holds, for every $X \in \mathcal{X}(M)$, $\alpha \in \mathcal{E}^1(M)$ and λ constant.

The metrics g and \tilde{g} are compatible if they are almost compatible and moreover the relation

$$g_\lambda^*(R_{X,Y}^\lambda \alpha) = g^*(R_{X,Y} \alpha) + \lambda \tilde{g}^*(\tilde{R}_{X,Y} \alpha)$$

holds, for every $\alpha \in \mathcal{E}^1(M)$, $X, Y \in \mathcal{X}(M)$ and λ constant.

If $R^\lambda = 0$ for all λ then g, \tilde{g} are said to form a flat-pencil of metrics.

The motivation for this definition comes from the theory of bi-Hamiltonian structures for equations of hydrodynamic type, i.e. for $(1+1)$ -dimensional evolution equations

$$\frac{\partial u^i}{\partial T} = M_j^i[u^r(X, T)] \frac{\partial u^j}{\partial X}.$$

The foundational result in this area is due to Dubrovin and Novikov [1]:

Theorem 2. *Given two functionals of hydrodynamic type (i.e. depending only on the fields $\{u^i\}$ and not their derivatives)*

$$F = \int_{S^1} f(u) dX, \quad G = \int_{S^1} g(u) dX$$

the bracket

$$\{F, G\} = \int_{S^1} \frac{\delta F}{\delta u^i} \left[g^{ij} \frac{d}{dX} - g^{is} \Gamma_{sk}^j u_X^k \right] \frac{\delta G}{\delta u^j} dX$$

defines (in the non-degenerate case $\det[g^{ij}] \neq 0$) a Hamiltonian structure if and only if

- g^{ij} is symmetric, and so defines a (pseudo)-Riemannian metric;
- Γ_{ij}^k are the Christoffel symbols of the Levi-Civita connection of g ;
- the curvature tensor of g is identically zero.

Such a Hamiltonian structure is said to be of Dubrovin/Novikov type.

The concept of a bi-Hamiltonian structure was introduced by Magri [9]. Given two Hamiltonian structures $\{, \}_1$ and $\{, \}_2$ then one may define a new bracket

$$\{, \}_\lambda = \{, \}_1 + \lambda \{, \}_2.$$

Definition 3. *If $\{, \}_\lambda$ is a Hamiltonian structure for all λ then the brackets $\{, \}_1$ and $\{, \}_2$ define a bi-Hamiltonian structure.*

It follows immediately from the above definitions and Theorem 2 that one has a bi-Hamiltonian structure of Dubrovin/Novikov type if and only if the corresponding metrics form a flat-pencil. The existence of such a pencil on the manifold M results in a very rich geometric structure and leads, with various extra conditions, to M being endowed with the structure of a Frobenius manifold [2].

Definition 4. [2]. *M is a Frobenius manifold if a structure of a Frobenius algebra (i.e. a commutative, associative algebra with multiplication denoted by \cdot , a unity element e and an inner product \langle, \rangle satisfying the invariance condition $\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$) is specified on any tangent plane $T_p M$ at any point $p \in M$ smoothly depending on the point such that:*

- (i) *The invariant metric $\tilde{g} = \langle, \rangle$ is a flat metric on M ;*
- (ii) *The unity vector field e is covariantly constant with respect to the Levi-Civita connection $\tilde{\nabla}$ for the metric \tilde{g}*

$$\tilde{\nabla} e = 0;$$

(iii) The symmetric 3-tensor $c(X, Y, Z) = \tilde{g}(X \cdot Y, Z)$ be such that the tensor

$$(\tilde{\nabla}_W c)(X, Y, Z)$$

is totally symmetric;

(iv) A vector field E - the Euler vector field - must be determined on M such that

$$\tilde{\nabla}(\tilde{\nabla}E) = 0$$

and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric and by rescalings on the Frobenius algebras $T_p M$.

Generalizations of Dubrovin/Novikov structures were introduced by Ferapontov [4]. Such structures (originally obtained by applying the Dirac theory of constrained dynamical systems) are of the form

$$\{F, G\} = \int_{S^1} \frac{\delta F}{\delta u^i} \left[g^{ij} \frac{d}{dX} - g^{is} \Gamma_{sk}^j u_X^k + \sum_{\alpha} w_{\alpha q}^i u_X^q (\nabla^{\perp})^{-1} w_{\alpha q}^j u_X^q \right] \frac{\delta G}{\delta u^j} dX$$

where (g, Γ, w) must satisfy certain geometric conditions, the crucial difference being the presence of curvature. Here the w may be interpreted as Weingarten maps and ∇^{\perp} as a normal connection. Bi-Hamiltonian structures may be similarly defined. Definition 1, first introduced by Mokhov [11], ensures that a compatible pair of metrics will define a bi-Hamiltonian structure of this generalized type, usually called a non-local bi-hamiltonian structure. No further mention will be made in this paper of bi-Hamiltonian structures, though this was one of the original motivations to study the geometry of compatible metrics.

The aim of this paper is to study the geometric structures on a manifold endowed with two compatible metrics, and conversely, to study the geometric conditions required for two metrics to be compatible. The constructions and results will mirror those in [2], but it will turn out that many of the results require only almost compatibility or compatibility and not flatness.

The rest of the paper is laid out as follows. In Section 2 the condition on the pair (g, \tilde{g}) required for almost compatibility is derived. This condition, the vanishing of the Nijenhuis tensor constructed from the pair, appeared in [11]. The proof given here is shorter and coordinate free. It is included both for completeness and to fix the notions used in later sections. Section 3 contains the central result of the paper: the conditions required for an almost compatible pair of metrics to be compatible. These conditions are interpreted in terms of an algebraic structure on the cotangent bundle in Section 4. Again, the ideas follow Dubrovin [2], but the algebraic structure comes from compatibility, not the flatness of the pencil. The concept of an \mathfrak{F} -manifold and a weak \mathfrak{F} -manifold are introduced in Section 5. With this the connection between (weak) quasi-homogeneous pencils of metrics and (weak) \mathfrak{F} -manifolds can be made precise. Curvature properties are studied in Section 6. In particular, when both metrics are flat one recovers the results of [2].

F -manifolds were introduced by Hertling and Manin [8]. An application of our results is that any F -manifold (with Euler field), equipped with a (non-necessarily flat) invariant metric

$$\tilde{g}(a \cdot b, c) = \tilde{g}(a, b \cdot c),$$

will generate a pencil of compatible metrics. Thus large classes of examples may be derived from singularity theory. However, a weaker notion is sufficient to ensure the existence of a pencil of compatible metrics. It was shown [7] that the F -manifold condition is related to the total symmetry of the tensor $\tilde{\nabla}(\cdot)$. A weak \mathfrak{F} -manifold, which ensures the existence of compatible metrics and hence of non-local bi-Hamiltonian structures, requires only that the tensor $\tilde{\nabla}(\cdot)(X, Y, Z, E)$ is totally symmetric in X, Y and Z , where E is the Euler vector field. Thus all F -manifolds with compatible metrics are weak \mathfrak{F} -manifolds but not vice-versa. The different font is used to denote the fact that the definition of a weak \mathfrak{F} -manifold includes a metric while the definition of an F -manifold is metric independent. Our results from Sections 5 and 6 can be summarized in Table 1, where the vertical arrows denote 1 : 1 correspondences.

The origin of this paper was one of the authors' work on the induced structures on submanifolds of Frobenius manifolds [12, 13]. It is natural to consider conditions for metrics on a submanifold, induced from a compatible pencil of metrics in the ambient manifold, to be almost compatible and compatible. Such questions are studied in Sections 7 and 8.

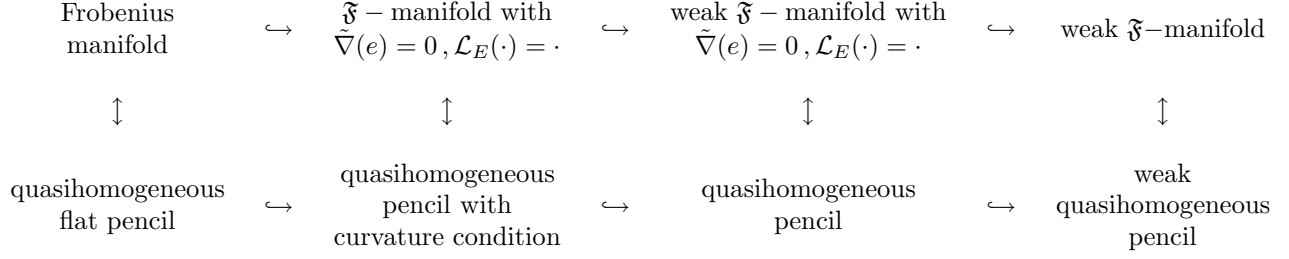


TABLE 1.

The Appendix contains a short proof that, in the semi-simple case, almost compatibility implies compatibility, a result originally obtained in [11]. Again, it is included here for completeness. Such a result is of interest in the study of semi-Hamiltonian hydrodynamic systems.

Various related results have already appeared in the literature, but with various additional assumptions, such as semi-simplicity or flatness of at least one of the metrics. Such distinguished additional structures simplify many of the calculations. Here no such simplifying assumptions are made. Finally, it should be straightforward to extend these results to the almost Frobenius structures introduced recently by Dubrovin [3] and studied further by Manin [10].

2. ALMOST COMPATIBLE METRICS

The following Theorem has been proved in [11]. We construct a new shorter proof which uses a coordinate free argument.

Theorem 5. *The metrics g and \tilde{g} are almost compatible if and only if the $(2,1)$ -tensor*

$$N_A(X, Y) := -[A(X), A(Y)] + A[A(X), Y] + A[X, A(Y)] - A^2[X, Y]$$

(with $A : TM \rightarrow TM$ defined by $A := \tilde{g}^*g$) vanishes identically.

Proof. We recall that the Levi-Civita connection ∇ of a metric g on M is determined on TM by the Koszul formula: for every $X, Y, Z \in \mathcal{X}(M)$,

$$\begin{aligned} g(\nabla_Y X, Z) &= \frac{1}{2} \{ X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z) \}. \end{aligned}$$

Let $X := g^*\alpha$ and $Z := g^*\gamma$. Then $X(g(Y, Z)) = g^*(\alpha, d(i_Y\gamma))$, $Z(g(X, Y)) = g^*(\gamma, d(i_Y\alpha))$ and

$$\begin{aligned} g([Y, Z], X) &= -g^*(L_Y(\alpha), \gamma) + Y(g^*(\alpha, \gamma)) \\ g([X, Y], Z) &= g^*(L_Y(\gamma), \alpha) - Y(g^*(\alpha, \gamma)). \end{aligned}$$

We deduce that the metrics g and \tilde{g} are almost compatible if and only if

$$(2) \quad g_\lambda([X_\lambda, Z_\lambda], Y) = g([X, Z], Y) + \lambda \tilde{g}([\tilde{X}, \tilde{Z}], Y)$$

holds, where

$$g_\lambda^*\alpha = X_\lambda; \quad g^*\alpha = X; \quad \tilde{g}^*\alpha = \tilde{X}$$

and

$$g_\lambda^*\gamma = Z_\lambda; \quad g^*\gamma = Z; \quad \tilde{g}^*\gamma = \tilde{Z}.$$

Since $g_\lambda^* = g^* + \lambda \tilde{g}^*$, $X_\lambda = X + \lambda \tilde{X}$ and $Z_\lambda = Z + \lambda \tilde{Z}$. Relation (2) becomes equivalent with

$$(3) \quad g_\lambda([X + \lambda \tilde{X}, Z + \lambda \tilde{Z}]) = g([X, Z]) + \lambda \tilde{g}([\tilde{X}, \tilde{Z}]).$$

Note that $\tilde{X} = A(X)$ and $\tilde{Z} = A(Z)$. Applying g_λ^* to both terms of the above equality and identifying the coefficients of λ we easily obtain the conclusion.

□

We shall use the following Proposition in our characterization of compatible metrics.

Proposition 6. *Suppose the metrics g and \tilde{g} are almost compatible. Then for every $\alpha, \gamma \in \mathcal{E}^1(M)$, the relation*

$$(4) \quad g^*(\tilde{\nabla}_{\tilde{g}^*\gamma}\alpha - \nabla_{\tilde{g}^*\gamma}\alpha) = \tilde{g}^*(\tilde{\nabla}_{g^*\gamma}\alpha - \nabla_{g^*\gamma}\alpha)$$

holds.

Proof. The Levi-Civita connection ∇ on T^*M is determined by the formula (see also the proof of Theorem 5)

$$2g^*(\nabla_Y\alpha, \beta) = -g^*(i_Y d\beta, \alpha) + g^*(i_Y d\alpha, \beta) + Yg^*(\alpha, \beta) - g([g^*\alpha, g^*\beta], Y).$$

Let $A := \tilde{g}^*g$ and define $A^*(\alpha)(X) = \alpha(AX)$ for every $\alpha \in \mathcal{E}^1(M)$ and $X \in \mathcal{X}(M)$. Then

$$(5) \quad \tilde{g}^*(\alpha, \beta) = g^*(\alpha, A^*\beta)$$

and the Koszul formula for \tilde{g} becomes

$$2g^*(\tilde{\nabla}_Y\alpha, A^*\beta) = -\tilde{g}^*(i_Y d\beta, \alpha) + \tilde{g}^*(i_Y d\alpha, \beta) + Y\tilde{g}^*(\alpha, \beta) - \tilde{g}([\tilde{g}^*\alpha, \tilde{g}^*\beta], Y).$$

It is now easy to see that

$$\begin{aligned} 2g^*(\nabla_Y\alpha - \tilde{\nabla}_Y\alpha, \beta) &= -g^*(i_Y d\beta, \alpha) + g^*(i_Y d(A^{-1*}\beta), A^*\alpha) - g([g^*\alpha, g^*\beta], Y) \\ &\quad + \tilde{g}([\tilde{g}^*\alpha, \tilde{g}^*A^{-1*}\beta], Y) \end{aligned}$$

and that

$$\begin{aligned} 2\tilde{g}^*(\nabla_Y\alpha - \tilde{\nabla}_Y\alpha, \beta) &= -g^*(i_Y d(A^*\beta), \alpha) + g^*(i_Y d\beta, A^*\alpha) - g([g^*\alpha, g^*A^*\beta], Y) \\ &\quad + \tilde{g}([\tilde{g}^*\alpha, \tilde{g}^*\beta], Y). \end{aligned}$$

In order to prove relation (4) we need to show

$$(6) \quad g^*(\tilde{\nabla}_{\tilde{g}^*\gamma}\alpha - \nabla_{\tilde{g}^*\gamma}\alpha, \beta) = \tilde{g}^*(\tilde{\nabla}_{g^*\gamma}\alpha - \nabla_{g^*\gamma}\alpha, \beta)$$

for every $\alpha, \beta, \gamma \in \mathcal{E}^1(M)$. Let $g^*\gamma = X$. Then $\tilde{g}^*\gamma = A(X)$ and relation (6) becomes

$$\begin{aligned} -(d\beta)(AX, g^*\alpha) + d(A^{-1*}\beta)(AX, g^*A^*\alpha) - g([g^*\alpha, g^*\beta], AX) + \tilde{g}([\tilde{g}^*\alpha, \tilde{g}^*A^{-1*}\beta], AX) \\ = -d(A^*\beta)(X, g^*\alpha) + d\beta(X, g^*A^*\alpha) - g([g^*\alpha, g^*A^*\beta], X) \\ + \tilde{g}([\tilde{g}^*\alpha, \tilde{g}^*\beta], X). \end{aligned}$$

Since $\tilde{g}^*A^{-1*} = g^*$ and $g^*A^* = \tilde{g}^*$ (see relation (5)), this is equivalent to

$$\begin{aligned} &\beta([AX, Z] - A^{-1}[AX, AZ] - A[X, Z] + [X, AZ]) \\ &\quad - g([Z, g^*\beta], AX) + \tilde{g}([AZ, g^*\beta], AX) \\ &\quad + g([Z, \tilde{g}^*\beta], X) - \tilde{g}([AZ, \tilde{g}^*\beta], X) \\ &= 0, \end{aligned}$$

where $Z := g^*\alpha$. Let $Y := g^*\beta$. The almost compatibility property of g and \tilde{g} implies that the first row of the above expression is zero and then the above expression reduces to

$$(7) \quad -g([Z, Y], AX) + \tilde{g}([AZ, Y], AX) + g([Z, AY], X) - \tilde{g}([AZ, AY], X) = 0.$$

Using $\tilde{g}(X, AY) = \tilde{g}(Y, AX) = g(X, Y)$, relation (7) becomes $\tilde{g}(N_A(Y, Z), X) = 0$ which is obviously true since $N_A = 0$.

□

3. COMPATIBLE METRICS

Theorem 7. *Suppose the metrics g and \tilde{g} are almost compatible. The following statements are equivalent:*

1. *The metrics g and \tilde{g} are compatible.*
2. *For every $\alpha, \beta \in \mathcal{E}^1(M)$ and $X, Y \in \mathcal{X}(M)$, the relation*

$$(8) \quad g^*(\tilde{\nabla}_Y \alpha - \nabla_Y \alpha, \tilde{\nabla}_X \beta - \nabla_X \beta) = g^*(\tilde{\nabla}_X \alpha - \nabla_X \alpha, \tilde{\nabla}_Y \beta - \nabla_Y \beta)$$

holds.

3. *For every $\alpha, \beta \in \mathcal{E}^1(M)$ and $X, Y \in \mathcal{X}(M)$, the relation*

$$(9) \quad \tilde{g}^*(\tilde{\nabla}_Y \alpha - \nabla_Y \alpha, \tilde{\nabla}_X \beta - \nabla_X \beta) = \tilde{g}^*(\tilde{\nabla}_X \alpha - \nabla_X \alpha, \tilde{\nabla}_Y \beta - \nabla_Y \beta)$$

holds.

Proof. Note that if h is a pseudo-Riemannian metric on M with Levi-Civita connection ∇' , then its curvature R' can be written in the form

$$\begin{aligned} h^*(R'_{h^*(\gamma), X} \alpha, \beta) &= (h^* \gamma) (h^*(\nabla'_X \alpha, \beta)) - X (h^*(\nabla'_{h^* \gamma} \alpha, \beta)) - h^*(\nabla'_{[h^* \gamma, X]} \alpha, \beta) \\ &\quad + h^*(\nabla'_{h^*(\nabla'_X \beta)} \alpha, \gamma) + d\alpha(h^* \gamma, h^* \nabla'_X \beta) \\ &\quad - h^*(\nabla'_{h^*(\nabla'_X \alpha)} \beta, \gamma) - d\beta(h^* \gamma, h^* \nabla'_X \alpha), \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathcal{E}^1(M)$ and $X \in \mathcal{X}(M)$. We shall use this observation for $h := g, \tilde{g}, g_\lambda$. Identifying the coefficients of λ in the compatibility condition

$$g_\lambda^*(R_{g_\lambda^*(\gamma), X}^\lambda \alpha, \beta) = g^*(R_{g_\lambda^*(\gamma), X} \alpha, \beta) + \lambda \tilde{g}^*(\tilde{R}_{g_\lambda^*(\gamma), X} \alpha, \beta)$$

and using relation (1), we see that g and \tilde{g} are compatible if and only if the expression

$$\begin{aligned} E_{\alpha, \beta, \gamma, X} &:= g^*(\nabla_X \alpha, \nabla_{\tilde{g}^* \gamma} \beta) + \tilde{g}^*(\tilde{\nabla}_X \alpha, \tilde{\nabla}_{g^* \gamma} \beta) - g^*(\nabla_{\tilde{g}^* \gamma} \alpha, \nabla_X \beta) - \tilde{g}^*(\tilde{\nabla}_{g^* \gamma} \alpha, \tilde{\nabla}_X \beta) \\ &\quad + \tilde{g}^*(\tilde{\nabla}_X \beta, \nabla_{g^* \gamma} \alpha) - \tilde{g}^*(\tilde{\nabla}_X \alpha, \nabla_{g^* \gamma} \beta) + g^*(\tilde{\nabla}_{g^* \gamma} \alpha, \nabla_X \beta) - g^*(\nabla_X \alpha, \tilde{\nabla}_{g^* \gamma} \beta) \end{aligned}$$

is zero, for every $\alpha, \beta, \gamma \in \mathcal{E}^1(M)$ and $X \in \mathcal{X}(M)$. Using Proposition 6, we notice that

$$\begin{aligned} E_{\alpha, \beta, \gamma, X} &= g^*(\nabla_X \alpha, \nabla_{\tilde{g}^* \gamma} \beta - \tilde{\nabla}_{\tilde{g}^* \gamma} \beta) + g^*(\tilde{\nabla}_{\tilde{g}^* \gamma} \alpha - \nabla_{\tilde{g}^* \gamma} \alpha, \nabla_X \beta) \\ &\quad + \tilde{g}^*(\tilde{\nabla}_X \beta, \nabla_{g^* \gamma} \alpha - \tilde{\nabla}_{g^* \gamma} \alpha) + \tilde{g}^*(\tilde{\nabla}_X \alpha, \tilde{\nabla}_{g^* \gamma} \beta - \nabla_{g^* \gamma} \beta) \\ &= \tilde{g}^*(\tilde{\nabla}_{g^* \gamma} \alpha - \nabla_{g^* \gamma} \alpha, \nabla_X \beta - \tilde{\nabla}_X \beta) \\ &\quad + \tilde{g}^*(\tilde{\nabla}_{g^* \gamma} \beta - \nabla_{g^* \gamma} \beta, \tilde{\nabla}_X \alpha - \nabla_X \alpha) \end{aligned}$$

and also

$$\begin{aligned} E_{\alpha, \beta, \gamma, X} &= g^*(\tilde{\nabla}_{\tilde{g}^* \gamma} \beta - \nabla_{\tilde{g}^* \gamma} \beta, \tilde{\nabla}_X \alpha - \nabla_X \alpha) \\ &\quad + g^*(\tilde{\nabla}_{\tilde{g}^* \gamma} \alpha - \nabla_{\tilde{g}^* \gamma} \alpha, \nabla_X \beta - \tilde{\nabla}_X \beta). \end{aligned}$$

The Theorem is proved. □

4. MULTIPLICATION ON T^*M

In this Section we show that the (almost) compatibility condition can also be formulated in terms of a multiplication “ \circ ” on T^*M . This multiplication has been used and studied in [2], when the metrics are flat. We here extend this study to the more general case of compatible metrics, not necessarily flat.

By a multiplication “ \circ ” on a vector bundle V we mean a bundle map

$$\circ : V \times V \rightarrow V.$$

The idea of defining a multiplication on the tangent bundle dates back to Vaisman [15]. Here a multiplication on the cotangent bundle is required [2].

Definition 8. *An arbitrary pair of metrics (g, \tilde{g}) on M defines a multiplication*

$$(10) \quad \alpha \circ \beta := \nabla_{g^* \alpha}(\beta) - \tilde{\nabla}_{g^* \alpha}(\beta)$$

on T^*M .

Note that, in general, the multiplications determined by (g, \tilde{g}) and (\tilde{g}, g) do not coincide. The next Proposition is a rewriting of the relations (2.5) and (2.6) of [2].

Proposition 9. *For every $\alpha, \beta, \gamma \in \mathcal{E}^1(M)$, the following relation holds:*

$$(11) \quad g^*(\alpha \circ \beta, \gamma) = g^*(\alpha, \gamma \circ \beta).$$

If g and \tilde{g} are almost compatible, then also

$$(12) \quad \tilde{g}^*(\alpha \circ \beta, \gamma) = \tilde{g}^*(\alpha, \gamma \circ \beta).$$

Proof. Relation (11) is a consequence of the torsion free property of the connections ∇ and $\tilde{\nabla}$:

$$\begin{aligned} g^*(\alpha \circ \beta, \gamma) - g^*(\alpha, \gamma \circ \beta) &= g^*(\nabla_{g^* \alpha} \beta - \tilde{\nabla}_{g^* \alpha} \beta, \gamma) - g^*(\nabla_{g^* \gamma} \beta - \tilde{\nabla}_{g^* \gamma} \beta, \alpha) \\ &= \nabla_{g^* \alpha}(\beta)(g^* \gamma) - \tilde{\nabla}_{g^* \alpha}(\beta)(g^* \gamma) - \nabla_{g^* \gamma}(\beta)(g^* \alpha) \\ &\quad + \tilde{\nabla}_{g^* \gamma}(\beta)(g^* \alpha) \\ &= d\beta(g^* \alpha, g^* \gamma) + d\beta(g^* \gamma, g^* \alpha) \\ &= 0. \end{aligned}$$

Suppose now that g and \tilde{g} are almost compatible. Relation (4) of Proposition 6 can be written as

$$\tilde{g}^*(\alpha \circ \beta, \gamma) = g^*(\nabla_{\tilde{g}^* \alpha} \beta - \tilde{\nabla}_{\tilde{g}^* \alpha} \beta, \gamma).$$

It follows that

$$\begin{aligned} \tilde{g}^*(\alpha \circ \beta, \gamma) - \tilde{g}^*(\alpha, \gamma \circ \beta) &= \nabla_{\tilde{g}^* \alpha}(\beta)(g^* \gamma) - \tilde{\nabla}_{\tilde{g}^* \alpha}(\beta)(g^* \gamma) \\ &\quad - \nabla_{g^* \gamma}(\beta)(\tilde{g}^* \alpha) + \tilde{\nabla}_{g^* \gamma}(\beta)(\tilde{g}^* \alpha) \\ &= d\beta(\tilde{g}^* \alpha, g^* \gamma) + d\beta(g^* \gamma, \tilde{g}^* \alpha) \\ &= 0. \end{aligned}$$

□

The following Proposition is a reformulation of Theorem 7 and generalizes equation (2.7) of [2].

Proposition 10. *Suppose that the metrics g and \tilde{g} are almost compatible. Then they are compatible if and only if the relation*

$$(13) \quad (\beta \circ \gamma) \circ \alpha = (\beta \circ \alpha) \circ \gamma$$

holds, for every $\alpha, \beta, \gamma \in \mathcal{E}^1(M)$.

Proof. Relation (9) is equivalent with

$$\tilde{g}^*(\lambda \circ \alpha, \beta \circ \gamma) = \tilde{g}^*(\lambda \circ \gamma, \beta \circ \alpha),$$

or, using (12), with

$$\tilde{g}^*(\lambda, (\beta \circ \gamma) \circ \alpha) = \tilde{g}^*(\lambda, (\beta \circ \alpha) \circ \gamma).$$

Since \tilde{g} is non-degenerate, the conclusion follows. □

The following Lemma relates, in a nice way, the curvatures of g and \tilde{g} with “ \circ ”-multiplication. We state it for completeness and we leave its proof to the reader.

Lemma 11. *Let (g, \tilde{g}) be an arbitrary pair of metrics on M , with corresponding multiplication \circ on T^*M . Then*

$$\begin{aligned} R_{g^*\alpha, g^*\beta}(\gamma) &= \tilde{R}_{g^*\alpha, g^*\beta}(\gamma) + \tilde{\nabla}_{g^*\alpha}(\circ)(\beta, \delta) - \tilde{\nabla}_{g^*\beta}(\circ)(\alpha, \delta) \\ &\quad + \alpha \circ (\beta \circ \delta) - (\alpha \circ \beta) \circ \delta - \beta \circ (\alpha \circ \delta) + (\beta \circ \alpha) \circ \delta \end{aligned}$$

for every $\alpha, \beta, \gamma, \delta \in \mathcal{E}^1(M)$.

Note that if $R = \tilde{R} = 0$ then one may integrate the equation

$$\tilde{\nabla}_{g^*\alpha}(\circ)(\beta, \delta) - \tilde{\nabla}_{g^*\beta}(\circ)(\alpha, \delta) = 0$$

in terms of potential functions. The remaining part of the equation - the defining condition for a Vinberg or pre-Lie algebra - gives a differential equation for the potential functions. The integrability of this differential equation was established by Ferapontov [5] and Mokhov [11].

5. QUASI-HOMOGENEOUS PENCIL OF METRICS AND WEAK \mathfrak{F} -MANIFOLDS

We now turn our attention to the case of (weak) quasi-homogeneous pencils and the parallel notion of (weak) \mathfrak{F} -manifolds. The aim of this Section is to prove the last two vertical 1-1 correspondences in Table 1 of the introduction.

Definition 12. (see [2]) *A pair (g, \tilde{g}) of compatible metrics on M is called a (regular) quasi-homogeneous pencil of degree d if the following two conditions are satisfied:*

- (1) *There is a smooth function f on M such that the vector fields $E := \text{grad}_g(f)$ and $e := \text{grad}_{\tilde{g}}(f)$ have the following properties:*

$$[e, E] = e; \quad L_E(g^*) = (d-1)g^*; \quad L_e(g^*) = \tilde{g}^*; \quad L_e(\tilde{g}^*) = 0.$$

- (2) *The operator $T(u) := \frac{d-1}{2}u + u(\tilde{\nabla}E)$ is an automorphism of T^*M .*

Remark: The following facts hold:

- (1) Since $L_e(\tilde{g}) = 0$ and $e = \text{grad}_{\tilde{g}}(f)$, e is $\tilde{\nabla}$ -parallel.
- (2) The conditions $L_E(g) = (1-d)g$ and $E = \text{grad}_g(f)$, easily imply that

$$(14) \quad \nabla_X(E) = \frac{1-d}{2}X,$$

for every $X \in \mathcal{X}(M)$. Also, E is a conformal Killing vector field of the metric \tilde{g} :

$$\begin{aligned} L_E(\tilde{g}^*) &= L_E L_e(g^*) = L_e L_E(g^*) + L_{[E, e]}(g^*) \\ &= (d-1)L_e(g^*) - L_e(g^*) = (d-2)\tilde{g}^*. \end{aligned}$$

A consequence of relation (14) is the following Proposition, which justifies Definition 14.

Proposition 13. *For every $u \in \mathcal{E}^1(M)$, $T(u) = df \circ u$.*

Proof. This is just a simple calculation: for every $X \in \mathcal{X}(M)$,

$$\begin{aligned} (df \circ u)(X) &= \nabla_{g^*}(df)(u)(X) - \tilde{\nabla}_{g^*}(df)(u)(X) \\ &= \nabla_E(u)(X) - \tilde{\nabla}_E(u)(X) \\ &= \nabla_E(u)(X) - E(u(X)) + u(\tilde{\nabla}_E X) \\ &= -u(\nabla_X E) + u(\tilde{\nabla}_X E) \\ &= \frac{d-1}{2}u(X) + u(\tilde{\nabla}_X E) \end{aligned}$$

where in the last equality we have used relation (14). □

Definition 14. *A pair of compatible metrics (g, \tilde{g}) is a (regular) weak quasi-homogeneous pencil of bi-degree (d, D) if the following two conditions are satisfied:*

- (1) *There is a vector field E on M with the properties: $L_E(g) = (1 - d)g$, $L_E(\tilde{g}) = D\tilde{g}$.*
- (2) *The operator $T(u) := \frac{d-1}{2}u + u(\tilde{\nabla}E)$ is an automorphism of T^*M and $T(u) = g(E) \circ u$, for every $u \in \mathcal{E}^1(M)$.*

Note that, any quasi-homogeneous pencil of degree d is also weak quasi-homogeneous of bi-degree $(d, 2 - d)$.

We now introduce the parallel notions of (weak) \mathfrak{F} -manifolds:

Definition 15. *A manifold M with a multiplication “ \cdot ” on its tangent bundle, a vector field E and a metric \tilde{g} is called an \mathfrak{F} -manifold if the following conditions are satisfied:*

- (1) *the multiplication “ \cdot ” is associative, commutative and has unity e ;*
- (2) *the vector field E (called the Euler vector field) admits an inverse E^{-1} with respect to the multiplication “ \cdot ”, satisfies $L_E(\cdot) = k\cdot$, $L_E(\tilde{g}) = D\tilde{g}$ and the operator $T(u) := \frac{D+k}{2}u - \tilde{g}(\tilde{\nabla}_{\tilde{g}^*(u)}E)$ is an automorphism of T^*M .*
- (3) *the metric \tilde{g} is “ \cdot ”-invariant: $\tilde{g}(X \cdot Y, Z) = \tilde{g}(X, Y \cdot Z)$, for every $X, Y, Z \in \mathcal{X}(M)$.*
- (4) *the $(4, 0)$ -tensor $\tilde{\nabla}(\cdot)$ of M defined by*

$$\tilde{\nabla}(\cdot)(X, Y, Z, V) := \tilde{g} \left(\tilde{\nabla}_X(\cdot)(Y, Z), V \right)$$

is symmetric in all its arguments.

Remark: By a result of Hertling [7], all \mathfrak{F} -manifolds (originally defined in [8]) are F -manifolds, i.e. the multiplication “ \cdot ” satisfies

$$L_{X \cdot Y}(\cdot) = Y \cdot L_X(\cdot) + X \cdot L_Y(\cdot)$$

for every $X, Y \in \mathcal{X}(M)$ and also the 1-form $g(e)$ is closed. The different typeface is used to denote the additional structures not present in the definition of an F -manifold.

Definition 16. *A weak \mathfrak{F} -manifold satisfies all the conditions of an \mathfrak{F} -manifold except (4), which is replaced by the weaker condition:*

$$(15) \quad \tilde{\nabla}(\cdot)(X, Y, Z, E) = \tilde{\nabla}(\cdot)(E, X, Y, Z)$$

for every $X, Y, Z \in \mathcal{X}(M)$, where E is the Euler vector field.

The tensor $\tilde{\nabla}(\cdot)$ is automatically symmetric in the last three variables, this following from the invariance property of the metric.

5.1. From weak \mathfrak{F} -manifolds to (weak) quasi-homogeneous pencils. We shall now prove the following theorem.

Theorem 17. *Let (M, \cdot, \tilde{g}, E) be a weak \mathfrak{F} -manifold with $L_E(\tilde{g}) = D\tilde{g}$, $L_E(\cdot) = k\cdot$ and identity e . Define the metric g by $g^*\tilde{g} = E \cdot \cdot$. The following facts hold:*

1. *The pair (g, \tilde{g}) is a weak quasi-homogeneous pencil of bi-degree $(1 + k - D, D)$.*
2. *If e is $\tilde{\nabla}$ -parallel and $k = 1$, then in a neighborhood of any point of M the pair (g, \tilde{g}) is a quasi-homogeneous pencil of degree $2 - D$. If moreover M is simply connected the pair (g, \tilde{g}) is a global quasi-homogeneous pencil.*

We divide the proof into several steps.

Lemma 18. *Let (M, \cdot, \tilde{g}, E) be a weak \mathfrak{F} -manifold. Then $N_E = 0$.*

Proof. The torsion free property of $\tilde{\nabla}$ implies that

$$\begin{aligned} N_E(X, Y) &= -\tilde{\nabla}_{E \cdot X}(E) \cdot Y - \tilde{\nabla}_{E \cdot X}(\cdot)(E, Y) + \tilde{\nabla}_{E \cdot Y}(E) \cdot X \\ &\quad + \tilde{\nabla}_{E \cdot Y}(\cdot)(E, X) - E \cdot X \cdot \tilde{\nabla}_Y(E) - E \cdot \tilde{\nabla}_Y(\cdot)(E, X) \\ &\quad + E \cdot Y \cdot \tilde{\nabla}_X(E) + E \cdot \tilde{\nabla}_X(\cdot)(E, Y), \end{aligned}$$

for every $X, Y \in \mathcal{X}(M)$. From relation (15) and the commutativity and associativity of “ \cdot ”, we see that

$$\begin{aligned} N_{E\cdot}(X, Y) &= -\tilde{\nabla}_{E\cdot X}(E) \cdot Y - \tilde{\nabla}_E(\cdot)(E \cdot X, Y) + \tilde{\nabla}_{E\cdot Y}(E) \cdot X \\ &\quad + \tilde{\nabla}_E(\cdot)(X, E \cdot Y) - E \cdot X \cdot \tilde{\nabla}_Y(E) + E \cdot Y \cdot \tilde{\nabla}_X(E) \\ &= L_E(E \cdot X) \cdot Y + E \cdot X \cdot L_E(Y) - X \cdot L_E(E \cdot Y) - E \cdot Y \cdot L_E(X) \\ &= 0, \end{aligned}$$

where in the last equality we have used $L_E(\cdot) = k \cdot$.

□

Proposition 19. *Let “ \cdot ” be an associative, commutative, with identity multiplication on TM , \tilde{g} a “ \cdot ”-invariant metric on M and E a vector field on M which satisfies $L_E(\cdot) = k \cdot$ and $L_E(\tilde{g}) = D\tilde{g}$, for D, k constant. Suppose that $E \cdot$ is an automorphism of TM . Define a new metric g on M by $g^* \tilde{g} = E \cdot$. If $E^b := \tilde{g}(E)$, then*

$$\begin{aligned} 2g^*(\nabla_{\tilde{g}^* \gamma} \alpha - \tilde{\nabla}_{\tilde{g}^* \gamma} \alpha, \beta) &= \tilde{g} \left((D+k)\alpha - 2\tilde{\nabla}_{\tilde{g}^* \alpha}(E^b), \gamma \cdot \beta \right) - \tilde{\nabla}(\cdot)(E^b, \alpha, \gamma, \beta) \\ &\quad + \tilde{\nabla}(\cdot)(\gamma, \beta, E^b, \alpha) - \tilde{\nabla}(\cdot)(\alpha, \beta, E^b, \gamma) \\ &\quad + \tilde{\nabla}(\cdot)(\beta, E^b, \alpha, \gamma) + \gamma \left(N_{E\cdot}(\tilde{g}^* \alpha, \tilde{g}^* \beta) \cdot E^{-1} \right) \end{aligned}$$

for every $\alpha, \beta, \gamma \in \mathcal{E}^1(M)$. In particular,

$$(16) \quad \nabla_{\tilde{g}^* \gamma} \alpha - \tilde{\nabla}_{\tilde{g}^* \gamma} \alpha = \frac{1}{2} \left((D+k)\alpha - 2\tilde{\nabla}_{\tilde{g}^* \alpha}(E^b) \right) \cdot (E^b)^{-1} \cdot \gamma$$

if and only if relation (15) is satisfied.

Proof. The multiplication “ \cdot ” on TM together with the metric \tilde{g} induce a multiplication on T^*M . For $Y \in \mathcal{X}(M)$, let $Y^b := \tilde{g}(Y)$. From the proof of Proposition 6 (with g and \tilde{g} interchanged), we obtain

$$\begin{aligned} 2g^*(\nabla_Y \alpha - \tilde{\nabla}_Y \alpha, \beta) &= d(\beta \cdot E^b)(Y, \tilde{g}^* \alpha) - d\beta(Y, \tilde{g}^*(E^b \cdot \alpha)) \\ &\quad + \tilde{g}(E \cdot [\tilde{g}^* \alpha, \tilde{g}^* \beta] - [E \cdot \tilde{g}^* \alpha, \tilde{g}^* \beta], Y) \\ &\quad + \tilde{g}(N_{E\cdot}(\tilde{g}^* \alpha, \tilde{g}^* \beta), E^{-1} \cdot Y), \end{aligned}$$

since in our case $A^*(\alpha) = \alpha \cdot E^b$. Let

$$E_1 := d(\beta \cdot E^b)(Y, \tilde{g}^* \alpha) - d\beta(Y, \tilde{g}^*(E^b \cdot \alpha))$$

and

$$E_2 = \tilde{g}(E \cdot [\tilde{g}^* \alpha, \tilde{g}^* \beta] - [E \cdot \tilde{g}^* \alpha, \tilde{g}^* \beta], Y).$$

Then

$$\begin{aligned} E_1 &= \tilde{\nabla}_Y(\beta \cdot E^b)(\tilde{g}^* \alpha) - \tilde{\nabla}_{\tilde{g}^* \alpha}(\beta \cdot E^b)(Y) - \tilde{\nabla}_Y(\beta) \left(\tilde{g}^*(E^b \cdot \alpha) \right) + \tilde{\nabla}_{\tilde{g}^*(E^b \cdot \alpha)}(\beta)(Y) \\ &= \tilde{g}^* \left(\beta, \tilde{\nabla}_Y(E^b) \cdot \alpha - \tilde{\nabla}_{\tilde{g}^* \alpha}(E^b) \cdot Y^b \right) + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot E}(\beta)(Y) - \tilde{\nabla}_{\tilde{g}^* \alpha}(\beta)(Y \cdot E) \\ &\quad + \tilde{\nabla}(\cdot)(Y^b, \beta, E^b, \alpha) - \tilde{\nabla}(\cdot)(\alpha, \beta, E^b, Y^b). \end{aligned}$$

On the other hand, since $\tilde{\nabla}$ is torsion free and \tilde{g} is “ \cdot ”-invariant,

$$\begin{aligned}
E_2 &= \tilde{g}([\tilde{g}^*\alpha, \tilde{g}^*\beta], E \cdot Y) - \tilde{g}([E \cdot \tilde{g}^*\alpha, \tilde{g}^*\beta], Y) \\
&= \tilde{\nabla}_{\tilde{g}^*\alpha}(\beta)(E \cdot Y) - \tilde{\nabla}_{\tilde{g}^*\beta}(\alpha)(E \cdot Y) \\
&\quad - \tilde{\nabla}_{E \cdot \tilde{g}^*\alpha}(\beta)(Y) + \tilde{g}(\tilde{\nabla}_{\tilde{g}^*\beta}(E \cdot \tilde{g}^*\alpha), Y) \\
&= \tilde{\nabla}_{\tilde{g}^*\alpha}(\beta)(E \cdot Y) - \tilde{\nabla}_{E \cdot \tilde{g}^*\alpha}(\beta)(Y) \\
&\quad + \left(\tilde{\nabla}_{\tilde{g}^*\beta}(E^b) \cdot \alpha \right)(Y) + \tilde{\nabla}(\cdot)(\beta, E^b, \alpha, Y^b) \\
&= \tilde{\nabla}_{\tilde{g}^*\alpha}(\beta)(E \cdot Y) - \tilde{\nabla}_{E \cdot \tilde{g}^*\alpha}(\beta)(Y) \\
&\quad + \tilde{\nabla}_{\tilde{g}^*\beta}(E^b)(Y \cdot \tilde{g}^*\alpha) + \tilde{\nabla}(\cdot)(\beta, E^b, \alpha, Y^b) \\
&= \tilde{\nabla}_{\tilde{g}^*\alpha}(\beta)(E \cdot Y) - \tilde{\nabla}_{E \cdot \tilde{g}^*\alpha}(\beta)(Y) \\
&\quad + d(E^b)(\tilde{g}^*\beta, Y \cdot \tilde{g}^*\alpha) + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot Y}(E^b)(\tilde{g}^*\beta) \\
&\quad + \tilde{\nabla}(\cdot)(\beta, E^b, \alpha, Y^b).
\end{aligned}$$

We deduce that

$$\begin{aligned}
E_1 + E_2 &= \tilde{g}^* \left(\beta, \tilde{\nabla}_Y(E^b) \cdot \alpha - \tilde{\nabla}_{\tilde{g}^*\alpha}(E^b) \cdot Y^b - i_{\tilde{g}^*(\alpha) \cdot Y} d(E^b) + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot Y}(E^b) \right) \\
&\quad + \tilde{\nabla}(\cdot)(Y^b, \beta, E^b, \alpha) - \tilde{\nabla}(\cdot)(\alpha, \beta, E^b, Y^b) + \tilde{\nabla}(\cdot)(\beta, E^b, \alpha, Y^b).
\end{aligned}$$

Let $\tilde{g}^*\alpha = X$ and $\tilde{g}^*\beta = Z$. Using $L_E(\tilde{g}) = D\tilde{g}$, $E_1 + E_2$ becomes

$$\begin{aligned}
&\tilde{g}(\tilde{\nabla}_Y(E) \cdot X, Z) - \tilde{g}(\tilde{\nabla}_X E, Y \cdot Z) - \tilde{g}(\tilde{\nabla}_{X \cdot Y} E, Z) + g(\tilde{\nabla}_Z E, X \cdot Y) + \tilde{g}(\tilde{\nabla}_{X \cdot Y} E, Z) \\
&\quad + \tilde{\nabla}(\cdot)(Y, Z, E, X) - \tilde{\nabla}(\cdot)(X, Z, E, Y) + \tilde{\nabla}(\cdot)(Z, E, X, Y) \\
&= D\tilde{g}(X \cdot Y, Z) + \tilde{g}(\tilde{\nabla}_Y(E) \cdot X, Z) - \tilde{g}(\tilde{\nabla}_X(E) \cdot Y, Z) - \tilde{g}(\tilde{\nabla}_{X \cdot Y} E, Z) \\
&\quad + \tilde{\nabla}(\cdot)(Z, E, X, Y) + \tilde{\nabla}(\cdot)(Y, Z, E, X) - \tilde{\nabla}(\cdot)(X, Z, E, Y).
\end{aligned}$$

It follows that

$$\begin{aligned}
2g^* \left(\nabla_Y X^b - \tilde{\nabla}_Y X^b, Z^b \right) &= \tilde{g} \left((DY + \tilde{\nabla}_Y E) \cdot X - \tilde{\nabla}_X(E) \cdot Y - \tilde{\nabla}_{X \cdot Y} E, Z \right) \\
&\quad + \tilde{\nabla}(\cdot)(Y, Z, E, X) - \tilde{\nabla}(\cdot)(X, Z, E, Y) + \tilde{\nabla}(\cdot)(Z, E, X, Y) \\
&\quad + \tilde{g}(N_E(X, Z), E^{-1} \cdot Y) \\
&= \tilde{g} \left((DY + \tilde{\nabla}_Y E) \cdot X + \tilde{\nabla}_E(X \cdot Y) - \tilde{\nabla}_E(X) \cdot Y, Z \right) \\
&\quad + \tilde{g} \left(-\tilde{\nabla}_E(Y) \cdot X - \tilde{\nabla}_X(E) \cdot Y - \tilde{\nabla}_{X \cdot Y}(E), Z \right) \\
&\quad + \tilde{\nabla}(\cdot)(Y, Z, E, X) - \tilde{\nabla}(\cdot)(X, Z, E, Y) + \tilde{\nabla}(\cdot)(Z, E, X, Y) \\
&\quad - \tilde{\nabla}(\cdot)(E, X, Y, Z) + \tilde{g}(N_E(X, Z), E^{-1} \cdot Y),
\end{aligned}$$

where in the second equality we have added the (identically zero) expression

$$\tilde{\nabla}_E(X \cdot Y) - \tilde{\nabla}_E(X) \cdot Y - X \cdot \tilde{\nabla}_E(Y) - \tilde{\nabla}_E(\cdot)(X, Y).$$

Using now $L_E(\cdot) = k \cdot$ we obtain

$$\begin{aligned}
2g^* \left(\nabla_Y X^b - \tilde{\nabla}_Y X^b, Z^b \right) &= \tilde{g} \left((DY + \tilde{\nabla}_Y E) \cdot X + L_E(X) \cdot Y + X \cdot L_E(Y) + kX \cdot Y, Z \right) \\
&\quad - \tilde{g} \left(\tilde{\nabla}_E(X) \cdot Y + \tilde{\nabla}_E(Y) \cdot X + \tilde{\nabla}_X(E) \cdot Y, Z \right) \\
&\quad + \tilde{\nabla}(\cdot)(Y, Z, E, X) - \tilde{\nabla}(\cdot)(X, Z, E, Y) + \tilde{\nabla}(\cdot)(Z, E, X, Y) \\
&\quad - \tilde{\nabla}(\cdot)(E, X, Y, Z) + \tilde{g} \left(N_{E \cdot}(X, Z), E^{-1} \cdot Y \right) \\
&= \tilde{g} \left((D + k)X \cdot Y - 2\tilde{\nabla}_X(E) \cdot Y, Z \right) + \tilde{\nabla}(\cdot)(Y, Z, E, X) \\
&\quad - \tilde{\nabla}(\cdot)(X, Z, E, Y) + \tilde{\nabla}(\cdot)(Z, E, X, Y) - \tilde{\nabla}(\cdot)(E, X, Y, Z) \\
&\quad + \tilde{g} \left(N_{E \cdot}(X, Z), E^{-1} \cdot Y \right).
\end{aligned}$$

The first statement of the Proposition is proved. We easily notice that

$$\begin{aligned}
&-\tilde{\nabla}(\cdot)(E, X, Y, Z) + \tilde{\nabla}(\cdot)(Y, Z, E, X) - \tilde{\nabla}(\cdot)(X, Z, E, Y) + \tilde{\nabla}(\cdot)(Z, E, X, Y) \\
&\quad + \tilde{g} \left(N_{E \cdot}(X, Z), E^{-1} \cdot Y \right) = 0
\end{aligned}$$

for every $X, Y, Z \in \mathcal{X}(M)$ is equivalent with relation (15) (exchange X and Z in the above equality, use the symmetry of $\tilde{\nabla}(\cdot)$ in the last three arguments and Lemma 18). \square

Proof of the theorem: Lemma 18 implies that the metrics g and \tilde{g} are almost compatible. The compatibility condition (9) is trivially satisfied using relation (16) and the “ \cdot ”-invariance of \tilde{g} . Since $g^* \tilde{g} = E \cdot$ and E is the Euler vector field, $L_E(g^*) \tilde{g} + Dg^* \tilde{g} = k g^* \tilde{g}$ or $L_E(g^*) = (k - D)g^*$. Relation (16) can be written in the form $g \tilde{g}^*(\gamma) \circ \alpha = T(\alpha) \cdot (E^b)^{-1} \cdot \gamma$, where T is precisely the operator associated to the pair (g, \tilde{g}) as in Definition 14. For $\gamma := \tilde{g}(E)$ we obtain $g(E) \circ \alpha = T(\alpha)$. It follows that (g, \tilde{g}) is a weak quasi-homogeneous pencil of bi-degree $(1 + k - D, D)$. The first statement is proved.

Suppose now that $k = 1$ and $\tilde{\nabla}(e) = 0$. In particular, $L_e(\tilde{g}) = 0$. Since $\tilde{\nabla}$ is torsion free, $d\tilde{g}(e) = 0$ and at least locally there is a smooth function f such that $\tilde{g}(e) = df$. Since $g^* \tilde{g} = E \cdot$, $E = g^* \tilde{g}(e) = \text{grad}_g(f)$. Also, $[e, E] = e$ because $k = 1$. In order to prove that (g, \tilde{g}) is a quasi-homogeneous pencil (of degree $2 - D$), we still need to show that $L_e(g^*) = \tilde{g}^*$. For this, consider the equality (which follows from $g^* \tilde{g} = E \cdot$)

$$g^*(\alpha, \beta) = \tilde{g}^*(\alpha, \beta \cdot E^b)$$

and take its Lie derivative in the direction of e . From $[e, E] = e$ and $L_e(\tilde{g}) = 0$ we get

$$L_e(g^*)(\alpha, \beta) = \tilde{g}^* \left(\alpha, L_e(\cdot)(\beta, E^b) \right) + \tilde{g}^*(\alpha, \beta).$$

On the other hand, from $\tilde{\nabla}(e) = 0$ and relation (15), we easily see that

$$\begin{aligned}
L_e(\cdot)(X, E) &= -[E \cdot X, e] + [X, e] \cdot E - X \\
&= \tilde{\nabla}_e(E \cdot X) + [X, e] \cdot E - X \\
&= \tilde{\nabla}_e(E) \cdot X + \tilde{\nabla}_E(\cdot)(e, X) + E \cdot \tilde{\nabla}_e(X) \\
&\quad + [X, e] \cdot E - X = 0,
\end{aligned}$$

which implies that $L_e(\cdot)(\beta, E^b) = \tilde{g}(L_e(\cdot)(\tilde{g}^* \beta, E)) = 0$, because $L_e(\tilde{g}) = 0$. It follows that $L_e(g^*) = \tilde{g}^*$ and the Theorem is proved.

5.2. From quasi-homogeneous pencil of metrics to weak \mathfrak{F} -manifolds.

Theorem 20. *Let (g, \tilde{g}) be a weak quasi-homogeneous pencil as in Definition 14. Define a new multiplication $u \cdot v := u \circ T^{-1}(v)$ on T^*M and denote also by “ \cdot ” the induced multiplication on TM , using the metric \tilde{g} . The following statements hold:*

- (1) (M, \cdot, \tilde{g}, E) is a weak \mathfrak{F} -manifold with Euler vector field E and identity $e := \tilde{g}^* g(E)$. Moreover, $g^* \tilde{g} = E \cdot$.

(2) If (g, \tilde{g}) is a quasi-homogeneous pencil then e is $\tilde{\nabla}$ -flat and $L_E(\cdot) = \cdot$ on TM .

We divide the proof into several steps. We begin with the following Lemma:

Lemma 21. *Let (g, \tilde{g}) be a weak quasi-homogeneous pencil. The following facts hold:*

- (1) *The multiplication “ \cdot ” on T^*M is associative, commutative and has unity $g(E)$.*
- (2) *For every $\alpha, \beta \in \mathcal{E}^1(M)$, $g^*(\alpha, \beta) = (\alpha \cdot \beta)(E)$.*

Proof. We prove the first claim: the commutativity $T(u) \cdot T(v) = T(v) \cdot T(u)$ is equivalent with $T(u) \circ v = T(v) \circ u$ or with $(g(E) \circ u) \circ v = (g(E) \circ v) \circ u$, which follows from relation (13), the metrics g and \tilde{g} being compatible. Using the commutativity “ \cdot ”, the associativity $(u \cdot v) \cdot w = u \cdot (v \cdot w)$ is equivalent with $(v \cdot u) \cdot w = (v \cdot w) \cdot u$, which is again a consequence of relation (13) and the definition of “ \cdot ”. Relation $T(v) = g(E) \circ v$, for every $v := T^{-1}(u) \in T^*M$ becomes $g(E) \cdot u = u$, for every $u \in T^*M$. This clearly means that the multiplication “ \cdot ” of T^*M has unity $g(E)$.

The second claim is an application of the definitions and of relation (11):

$$\begin{aligned} g^*(T(\beta), \alpha) - (\alpha \circ \beta)(E) &= g^*(g(E) \circ \beta, \alpha) - (\alpha \circ \beta)(E) \\ &= g^*(g(E), \alpha \circ \beta) - (\alpha \circ \beta)(E) \\ &= 0. \end{aligned}$$

□

Proof of the Theorem: The multiplication “ \cdot ” being associative and commutative on T^*M , the induced multiplication “ \cdot ” on TM has the same properties and has the identity $e = \tilde{g}^*g(E)$. Since (g, \tilde{g}) is a weak quasi-homogeneous pencil, g and \tilde{g} are in particular almost compatible and relation (12) with β replaced by $T^{-1}(\beta)$ implies the “ \cdot ”-invariance of \tilde{g} . An immediate consequence of the “ \cdot ”-invariance of \tilde{g} is the relation $g^*\tilde{g} = E \cdot$: indeed, to show this we notice from the second part of Lemma 21 that $g^*(\alpha, \beta) = \tilde{g}^*(\alpha \cdot \beta, E^b) = \tilde{g}^*(\alpha \cdot E^b, \beta)$, which implies that $g^*\alpha = \tilde{g}^*(\alpha \cdot E^b)$, or, for $\alpha := \tilde{g}(X)$, $g^*\tilde{g}(X) = E \cdot X$. In order to prove that E satisfies the conditions of an Euler vector field, we notice first that $L_E(\tilde{\nabla}) = L_E(\nabla) = 0$ (since $L_E(g) = (1-d)g$, $L_E(\tilde{g}) = D\tilde{g}$ and d, D are constant), which imply in particular that $L_E(T) = 0$. This, together with $L_E(g^*) = (d-1)g^*$, imply that $L_E(\cdot) = (d-1)\cdot$ on T^*M or, using $L_E(\tilde{g}) = D\tilde{g}$, that $L_E(\cdot) = (d+D-1)\cdot$ on TM . We can now apply Proposition 19 to prove the weak \mathfrak{F} -manifold condition (15): the very definition of “ \cdot ”, “ \circ ” and the relation (14) imply that relation (16) is satisfied. Thus (M, \cdot, \tilde{g}, E) is a weak \mathfrak{F} -manifold and the first claim of the Theorem is proved.

The second claim of the Theorem is trivial: if $D = 2 - d$, then $L_E(\cdot) = \cdot$ on TM and moreover, the very definition of quasi-homogeneous pencils implies that e is $\tilde{\nabla}$ -flat.

6. CURVATURE PROPERTIES OF WEAK \mathfrak{F} -MANIFOLDS

In this Section we will prove the first two vertical 1-1 correspondences in Table 1 of the introduction. In particular, we show how Dubrovin’s correspondence between flat quasi-homogeneous pencils and Frobenius manifolds fits into our theory.

We begin with the following simple Lemma on conformal Killing vector fields on pseudo-Riemannian manifolds.

Lemma 22. *Let (M, \tilde{g}) be a pseudo-Riemannian manifold and $E \in \mathcal{X}(M)$ which satisfies $L_E(\tilde{g}) = D\tilde{g}$, with D constant. Then*

$$\tilde{g}(\tilde{R}_{Z,X}(E), Y) = g\left(\tilde{\nabla}_X(\tilde{\nabla}E)_Y, Z\right)$$

for every $X, Y, Z \in \mathcal{X}(M)$.

Proof. We take the covariant derivative with respect to Z of the equality

$$\tilde{g}(\tilde{\nabla}_X E, Y) + \tilde{g}(\tilde{\nabla}_Y E, X) = D\tilde{g}(X, Y),$$

we use $\tilde{\nabla}(\tilde{g}) = 0$ and then we take cyclic permutations of X, Y, Z . We obtain three relations. Substracting the second and the third relation from the first one and using the symmetries of pseudo-Riemannian curvature tensors, we easily obtain the result. □

Thus if $\tilde{R} = 0$, then E can be at most linear in the flat coordinates.

Theorem 23. *Let (M, \cdot, \tilde{g}, E) be a weak \mathfrak{F} -manifold as in Definition 16. Then for every $\alpha \in \mathcal{E}^1(M)$ and $X, Y \in \mathcal{X}(M)$, the following relation holds:*

$$\begin{aligned} R_{E \cdot X, E \cdot Y}(\alpha) - \tilde{R}_{E \cdot X, E \cdot Y}(\alpha) &= \tilde{\nabla}_{E \cdot X}(\cdot)(T(\alpha) \cdot (E^\flat)^{-1}, E^\flat \cdot Y^\flat) - \tilde{\nabla}_{E \cdot Y}(\cdot)(T(\alpha) \cdot (E^\flat)^{-1}, E^\flat \cdot X^\flat) \\ &\quad + \tilde{\nabla}_{E \cdot X}(T)(\alpha) \cdot Y^\flat - \tilde{\nabla}_{E \cdot Y}(T)(\alpha) \cdot X^\flat. \end{aligned}$$

In particular, M is an \mathfrak{F} -manifold if and only if

$$R_{E \cdot X, E \cdot Y}(\alpha) - \tilde{R}_{E \cdot X, E \cdot Y}(\alpha) = \tilde{\nabla}_{E \cdot X}(T)(\alpha) \cdot Y^\flat - \tilde{\nabla}_{E \cdot Y}(T)(\alpha) \cdot X^\flat,$$

for every $\alpha \in \mathcal{E}^1(M)$ and $X, Y \in \mathcal{X}(M)$.

Proof. Let $Q(\alpha) := T(\alpha) \cdot E^\flat^{-1}$. Using relation (16), we easily see that

$$\begin{aligned} R_{X, Y}(\alpha) - \tilde{R}_{X, Y}(\alpha) &= [\tilde{\nabla}_X(Q(\alpha)) - Q(\tilde{\nabla}_X \alpha) - Q(Q(\alpha) \cdot X^\flat)] \cdot Y^\flat \\ &\quad - [\tilde{\nabla}_Y(Q(\alpha)) - Q(\tilde{\nabla}_Y \alpha) - Q(Q(\alpha) \cdot Y^\flat)] \cdot X^\flat \\ &\quad + \tilde{\nabla}_X(\cdot)(Q(\alpha), Y^\flat) - \tilde{\nabla}_Y(\cdot)(Q(\alpha), X^\flat) \end{aligned}$$

which becomes, after replacing α with $T^{-1}(\alpha)$, X, Y with $E \cdot X, E \cdot Y$ respectively, the following relation:

$$\begin{aligned} R_{E \cdot X, E \cdot Y}(T^{-1} \alpha) - \tilde{R}_{E \cdot X, E \cdot Y}(T^{-1} \alpha) &= \tilde{\nabla}_{E \cdot X}(\cdot)(\alpha \cdot (E^\flat)^{-1}, E^\flat \cdot Y^\flat) \\ &\quad - \tilde{\nabla}_{E \cdot Y}(\cdot)(\alpha \cdot (E^\flat)^{-1}, E^\flat \cdot X^\flat) \\ &\quad + [\tilde{\nabla}_{E \cdot X}(\alpha \cdot (E^\flat)^{-1}) \cdot E^\flat + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot X}(E^\flat)] \cdot Y^\flat \\ &\quad - [\tilde{\nabla}_{E \cdot Y}(\alpha \cdot (E^\flat)^{-1}) \cdot E^\flat + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot Y}(E^\flat)] \cdot X^\flat \\ &\quad - Q(\tilde{\nabla}_{E \cdot X}(T^{-1} \alpha)) \cdot E^\flat \cdot Y^\flat + Q(\tilde{\nabla}_{E \cdot Y}(T^{-1} \alpha)) \cdot E^\flat \cdot X^\flat. \end{aligned}$$

Define now $\mathcal{A}(\alpha, X) := \tilde{\nabla}_{E \cdot X}(\alpha \cdot (E^\flat)^{-1}) \cdot E^\flat + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot X}(E^\flat)$. Using the fact that M is a weak \mathfrak{F} -manifold, we easily get

$$\begin{aligned} \mathcal{A}(\alpha, X) &= \tilde{\nabla}_{E \cdot X}(\alpha) - \tilde{\nabla}_{E \cdot X}(\cdot)(E^\flat, \alpha \cdot (E^\flat)^{-1}) - \alpha \cdot (E^\flat)^{-1} \cdot \tilde{\nabla}_{E \cdot X}(E^\flat) + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot X}(E^\flat) \\ &= \tilde{\nabla}_{E \cdot X}(\alpha) - \tilde{\nabla}_E(\cdot)(E^\flat \cdot X^\flat, \alpha \cdot (E^\flat)^{-1}) - \alpha \cdot (E^\flat)^{-1} \cdot \tilde{\nabla}_{E \cdot X}(E^\flat) + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot X}(E^\flat) \\ &= \tilde{\nabla}_{E \cdot X}(\alpha) - \tilde{\nabla}_E(\alpha \cdot X^\flat) + \tilde{\nabla}_E(E^\flat \cdot X^\flat) \cdot \alpha \cdot (E^\flat)^{-1} + E^\flat \cdot X^\flat \cdot \tilde{\nabla}_E(\alpha \cdot (E^\flat)^{-1}) \\ &\quad - \alpha \cdot (E^\flat)^{-1} \cdot \tilde{\nabla}_{E \cdot X}(E^\flat) + \tilde{\nabla}_{\tilde{g}^*(\alpha) \cdot X}(E^\flat) \\ &= \tilde{\nabla}_{E \cdot X}(\alpha) - \tilde{g}(L_E(\tilde{g}^*(\alpha) \cdot X)) + \tilde{g}(L_E(E \cdot X)) \cdot \alpha \cdot (E^\flat)^{-1} + E^\flat \cdot X^\flat \cdot \tilde{\nabla}_E(\alpha \cdot (E^\flat)^{-1}) \\ &= \tilde{\nabla}_{E \cdot X}(\alpha) + X^\flat \cdot h(\alpha), \end{aligned}$$

where in the last equality we have used $L_E(\cdot) = k \cdot$ on TM and $h := h(\alpha)$ is an expression which depends only on α . We thus obtain

$$\begin{aligned} R_{E \cdot X, E \cdot Y}(T^{-1} \alpha) - \tilde{R}_{E \cdot X, E \cdot Y}(T^{-1} \alpha) &= \tilde{\nabla}_{E \cdot X}(\cdot)(\alpha \cdot (E^\flat)^{-1}, E^\flat \cdot Y^\flat) - \tilde{\nabla}_{E \cdot Y}(\cdot)(\alpha \cdot (E^\flat)^{-1}, E^\flat \cdot X^\flat) \\ &\quad + [\tilde{\nabla}_{E \cdot X}(\alpha) - Q(\tilde{\nabla}_{E \cdot X}(T^{-1} \alpha)) \cdot E^\flat] \cdot Y^\flat \\ &\quad - [\tilde{\nabla}_{E \cdot Y}(\alpha) - Q(\tilde{\nabla}_{E \cdot Y}(T^{-1} \alpha)) \cdot E^\flat] \cdot X^\flat \end{aligned}$$

which easily implies the Theorem. \square

Corollary 24. *(see [2]) Let (g, \tilde{g}) be a quasi-homogeneous pencil on M . Suppose that $\tilde{R} = 0$. Then the corresponding weak \mathfrak{F} -manifold is a Frobenius manifold if and only if $R = 0$.*

Proof. This is just a consequence of Theorem 23 and Lemma 22: since $\tilde{R} = 0$, $\tilde{\nabla}(T) = 0$ and

$$R_{E \cdot X, E \cdot Y}(\alpha) = \tilde{\nabla}_{E \cdot X}(\cdot)(T(\alpha) \cdot (E^b)^{-1}, E^b \cdot Y^b) - \tilde{\nabla}_{E \cdot Y}(\cdot)(T(\alpha) \cdot (E^b)^{-1}, E^b \cdot X^b)$$

for every $X, Y \in \mathcal{X}(M)$ and $\alpha \in \mathcal{E}^1(M)$. It follows that $R = 0$ is equivalent to the total symmetry of $\tilde{\nabla}(\cdot)$. The other conditions of a Frobenius manifold are clearly satisfied. \square

7. COMPATIBLE METRICS AND SUBMANIFOLDS

Suppose now that the metrics g and \tilde{g} are compatible. Let h, \tilde{h} be the metrics induced by g, \tilde{g} on a submanifold N of M . We assume that h, \tilde{h}, h_λ are non-degenerate (although a theory of bi-Hamiltonian structures with degenerate metrics may be formulated [6, 14]). Let $A := \tilde{g}^*g$ and $B := \tilde{h}^*h$.

Notations: We shall use the following conventions:

- (1) $TN^{\perp g}$ ($TN^{\perp \tilde{g}}$ respectively) will denote the orthogonal complement of TN in TM , with respect to the metric g (respectively \tilde{g}).
- (2) For a sub-bundle V of TM , V^0 will denote the annihilator of V in T^*M .
- (3) With respect to the orthogonal decomposition $TM = TN \oplus TN^{\perp \tilde{g}}$ any tangent vector $X \in TM$ will be written as $X = X^t + X^n$.

The following Lemma can be easily proved and justifies Definition 26.

Lemma 25. *For every $X \in TN$, $B(X) = A(X)^t$. In particular, the following statements are equivalent:*

- (1) *For every $X \in TN$, $A(X) = B(X)$.*
- (2) *$A(TN) \subset TN$.*
- (3) *The orthogonal complement of TN with respect to the metrics g and \tilde{g} coincide.*

Definition 26. *The submanifold N of M is called distinguished if $A(TN) \subset TN$.*

For now on we shall restrict to the case when the submanifold N is distinguished and we shall denote by TN^\perp the orthogonal complement of TN in TM , with respect to the metric g or \tilde{g} . For $\alpha \in \mathcal{E}^1(N)$, we will denote by $\bar{\alpha} \in \mathcal{E}^1(M)$ its extension to TM , which is zero on TN^\perp .

It can be easily verified that the restrictions to N of the metrics g_λ generated by (g, \tilde{g}) coincide with the metrics h_λ generated by (h, \tilde{h}) . Since $A = B$ on TN , the metrics h and \tilde{h} are almost compatible. A natural problem which arises is to determine when they are compatible. For this, let D (respectively \tilde{D}) be the Levi-Civita connections of h and \tilde{h} . For every $\alpha \in \mathcal{E}^1(N)$ and $X \in \mathcal{X}(N)$ we decompose $\nabla_X \bar{\alpha}$ as

$$(17) \quad \nabla_X \bar{\alpha} = \nabla_X(\bar{\alpha})^t + S_X(\bar{\alpha})$$

according to the decomposition

$$T^*M = (TN^\perp)^0 + (TN)^0$$

of T^*M . Note that, via the identifications $T^*N \cong (TN^\perp)^0$ and $(TN^\perp)^* \cong (TN)^0$, $D_X(\alpha) = \nabla_X(\bar{\alpha})^t$ and that the map $S : TN \times T^*N \rightarrow (TN)^0$, defined by $S_X(\alpha) = S_X(\bar{\alpha})$, is the second fundamental form of the submanifold N of (M, g) . Similar facts hold for \tilde{D} and the second fundamental form \tilde{S} of the submanifold N of (M, \tilde{g}) .

Proposition 27. *The metrics h and \tilde{h} are compatible if and only if for every $\alpha, \beta \in (TN^\perp)^0$ and $X, Y \in TN$, the relation*

$$(18) \quad \tilde{g}^*(\tilde{S}_X \alpha - S_X \alpha, \tilde{S}_Y \beta - S_Y \beta) = \tilde{g}^*(\tilde{S}_Y \alpha - S_Y \alpha, \tilde{S}_X \beta - S_X \beta),$$

holds.

Proof. This is just a consequence of relation (9) and of the decomposition (17). \square

Remark: The compatibility condition on the metrics h and \tilde{h} can also be written in terms of the multiplication \circ from Definition 8. Denote by

$$\circ^t : T^*M \times T^*M \rightarrow (TN^\perp)^0$$

and

$$\circ^n : T^*M \times T^*M \rightarrow (TN)^0$$

the maps induced by \circ and the orthogonal projections $T^*M \rightarrow (TN^\perp)^0$, $T^*M \rightarrow (TN)^0$. Let \circ_N be the multiplication associated to the pair (h, \tilde{h}) , as in Definition 8. Then, for every $\alpha, \beta \in \mathcal{E}^1(N)$,

$$\alpha \circ_N \beta = D_{h^*\alpha}(\beta) - \tilde{D}_{h^*\alpha}(\beta) = \nabla_{g^*\bar{\alpha}}(\bar{\beta})^t - \nabla_{g^*\bar{\alpha}}(\bar{\beta})^t = \bar{\alpha} \circ^t \bar{\beta},$$

which implies $\alpha \circ_N \beta = \bar{\alpha} \circ^t \bar{\beta}$. It follows that h and \tilde{h} are compatible if and only if for every $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in (TN^\perp)^0$,

$$(\bar{\alpha} \circ^t \bar{\beta}) \circ^t \bar{\gamma} = (\bar{\alpha} \circ^t \bar{\gamma}) \circ^t \bar{\beta}.$$

8. SUBMANIFOLDS OF WEAK \mathfrak{F} -MANIFOLDS

In this Section we consider a weak \mathfrak{F} -manifold (M, \cdot, \tilde{g}, E) with $L_E(\tilde{g}) = D\tilde{g}$ and $L_E(\cdot) = k \cdot$. Let N be a submanifold of M which satisfies the following two properties:

- (1) For every X, Y in TN , $X \cdot Y$ belongs to TN .
- (2) The Euler vector field E satisfies

$$(19) \quad E \cdot TN \subset TN.$$

We note that any natural submanifold [12] of an \mathfrak{F} -manifold satisfies these conditions, but in our case the vector field E is not necessarily tangent to N along N . We shall denote by TN^\perp the orthogonal complement of TN in TM , with respect to the metric \tilde{g} and by $P : TM \rightarrow TN^\perp$ the orthogonal projection.

Lemma 28. *For every $X \in TN$ and $Y \in TM$, $X \cdot P(Y) = P(X \cdot Y)$.*

Proof. Since \tilde{g} is “ \cdot ”-invariant, the operator $X \cdot$ of TM is self-adjoint (with respect to the metric \tilde{g}). Since it preserves TN , it will preserve TN^\perp as well. The conclusion follows. \square

Recall now that on a weak \mathfrak{F} -manifold we have considered a second metric g , defined by $g^*\tilde{g} = E \cdot$.

Proposition 29. *The metrics g and \tilde{g} induce compatible metrics on N .*

Proof. The almost compatibility is obvious from relation (19). Let $P^* : T^*M \rightarrow (TN)^0$ be the orthogonal projection with respect to the metric \tilde{g}^* . Relation (16) together with Lemma 28 imply that, for every $\alpha, \beta \in (TN^\perp)^0$, $S_X\alpha - \tilde{S}_X\alpha = \frac{1}{2}P^*Q(\alpha) \cdot X^\flat$, where, we recall, $Q(\alpha) = \frac{1}{2} \left((D+k)\alpha - 2\tilde{\nabla}_{\tilde{g}^*\alpha}(E^\flat) \right) \cdot (E^\flat)^{-1}$. It is now obvious that relation (18) is satisfied. \square

APPENDIX: THE SEMI-SIMPLE CASE

Recall that we have associated to the pair (g, \tilde{g}) an endomorphism N_A of TM (see Theorem 5).

Definition 30. *The pair (g, \tilde{g}) is semi-simple if the eigenvalues of the tensor N_A are distinct in every point.*

Theorem 31. *Any semi-simple pair of almost compatible metrics is compatible.*

Proof. If (g, \tilde{g}) is a semi-simple pair, we can find coordinates (x_1, \dots, x_n) on M such that the tensor $A := \tilde{g}^*g$ is diagonal: $\tilde{g}^*dx_i = \lambda_i g^*dx_i$, for $i = \overline{1, n}$ and moreover, both g and \tilde{g} are diagonal: $g^*(dx_i, dx_j) = \delta_{ij}g^{ii}$; $\tilde{g}^*(dx_i, dx_j) = \delta_{ij}\tilde{g}^{ii}$, for $i, j = \overline{1, n}$, for some smooth functions λ_i, g^{ii} and \tilde{g}^{ii} . The almost compatibility condition implies that the functions λ_i depend only on the coordinate x_i (see [11]). Using the formula for the Cristoffel symbols in a chart and the fact that λ_i depend only on the coordinate x_i , we easily see that $\tilde{\Gamma}_{ik}^j - \Gamma_{ik}^j = 0$ for $i \neq k$ and every j . Then

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}} dx_j - \nabla_{\frac{\partial}{\partial x_i}} dx_j = - \left(\tilde{\Gamma}_{ii}^j - \Gamma_{ii}^j \right) dx_i$$

which implies

$$\begin{aligned}\tilde{g}^* \left(\tilde{\nabla}_{\frac{\partial}{\partial x_i}} dx_j - \nabla_{\frac{\partial}{\partial x_i}} dx_j, \tilde{\nabla}_{\frac{\partial}{\partial x_k}} dx_n - \nabla_{\frac{\partial}{\partial x_k}} dx_n \right) &= (\tilde{\Gamma}_{ii}^j - \Gamma_{ii}^j)(\tilde{\Gamma}_{kk}^n - \Gamma_{kk}^n) \tilde{g}^*(dx_i, dx_k) \\ &= \delta_{ik} \tilde{g}^{ii} (\tilde{\Gamma}_{ii}^j - \Gamma_{ii}^j) (\tilde{\Gamma}_{kk}^n - \Gamma_{kk}^n).\end{aligned}$$

This expression is obviously symmetric in i and k . □

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